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Variance of the Strehl Ratio of an Adaptive Optics System

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The variance σ_s^2 of the Strehl ratio of a reasonably well-corrected adaptive optics system is derived as a power series in the log-amplitude variance σ_l^2 and the residual phase error variance $\sigma_{\delta\phi}^2$. It is shown that, to leading order, the variance of the Strehl ratio is dependent on the first power of the log-amplitude variance, $(\sigma_l^2)^1$, of the incident optical field but only on the second power of the residual phase variance, $(\sigma_{\delta\phi}^2)^2$, of that field after adaptive optics correction, and on the first power of the product of the log-amplitude variance times the phase variance, $(\sigma_l^2\sigma_{\delta\phi}^2)^1$. As long as the adaptive optics correction is good enough to ensure that the variance of the residual phase, $\sigma_{\delta\phi}^2$ is significantly less than unity, then even for fairly small values of the log-amplitude variance σ_l^2, the value of the variance of the Strehl ratio, σ_s^2, will be dominated by the value of the log-amplitude variance.

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Variance of the Strehl ratio of an adaptive optics system

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The variance σ_S^2 of the Strehl ratio of a reasonably well-corrected adaptive optics system is derived as a power series in the log-amplitude variance σ_l^2 and the residual phase error variance $\sigma_{\delta\phi}^2$. It is shown that, to leading order, the variance of the Strehl ratio is dependent on the first power of the log-amplitude variance, $(\sigma_l^2)^1$, of the incident optical field but only on the second power of the residual phase variance, $(\sigma_{\delta\phi}^2)^2$, of that field after adaptive optics correction, and on the first power of the product of the log-amplitude variance times the phase variance, $(\sigma_l^2\sigma_{\delta\phi}^2)^1$. As long as the adaptive optics correction is good enough to ensure that the variance of the residual phase, $\sigma_{\delta\phi}^2$, is significantly less than unity, then even for fairly small values of the log-amplitude variance σ_l^2, the value of the variance of the Strehl ratio, σ_S^2, will be dominated by the value of the log-amplitude variance. [S0740-3232(98)02008-0]

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In studying the variability of the Strehl ratio of an adaptive optics system, we have derived an analytic expression for the variance of the Strehl ratio, σ_S^2, as a power series in the residual phase variance $\sigma_{\delta\phi}^2$ and the log-amplitude variance σ_l^2 in a regime where each of these two variances is significantly less than unity. We have been surprised to find that our result for σ_S^2 has no linear dependence on the residual phase error variance $\sigma_{\delta\phi}^2$. The consequence of this is that in general the value of the variance σ_S^2 of the Strehl ratio is dominated by the value of the incident optical field's log-amplitude variance σ_l^2 and not by the value $\sigma_{\delta\phi}^2$ of the variance (or mean square value) of the residual phase error after adaptive optics correction. While improving the adaptive optics performance, i.e., reducing the value of $\sigma_{\delta\phi}^2$, will improve the mean value of the Strehl ratio, $\langle S\rangle$, bringing it closer to unity, beyond some fairly modest level of performance the further improvement of the adaptive optics will not significantly reduce the variance of the Strehl ratio. At that point the value of the variance of the Strehl ratio is dominated by the pattern over the system's aperture of the incident field's optical intensity variations—by what we call the "random apodization" of the aperture.

We have considered a ground-based adaptive optics telescope system, a system looking upward and operating in the visible or the near infrared—for which case the log-amplitude variance will generally be of the order of $\sigma_l^2 = 0.01$ to $0.10\,Np^2$—and have concerned ourselves with the variability from instant to instant of the system's performance, the performance being measured by the Strehl ratio S. We have restricted our attention to the Strehl ratio, as this quantity serves as a surrogate for the less exactly defined concept of resolution[1] and as a surrogate for the concept of antenna gain. Our interest in the random variability from instant to instant has to do with applications, such as short-exposure imagery, for which its value at a particular *instant* that will determine the system user's assessment of the system's performance.

The value of S is set by both the residual phase error, i.e., by the (small) part of the turbulence-induced wave-front distortion that for various practical reasons the adaptive optics did not correct, and by the variations over the telescope's aperture of the intensity of the received optical signal.[2]

An analysis of the expected values of the first and second moments of the Strehl ratio, $\langle S\rangle$ and $\langle S^2\rangle$, respectively, shows that these quantities have values given by the equations

$$\langle S\rangle = \frac{\iint \mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)\exp(F_1)}{\iint \mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{1}$$

$$\langle S^2\rangle = \frac{\iiiint \mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2\mathrm{d}\mathbf{r}_3\mathrm{d}\mathbf{r}_4 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)W(\mathbf{r}_4)\exp(F_2)}{\iiiint \mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2\mathrm{d}\mathbf{r}_3\mathrm{d}\mathbf{r}_4 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)W(\mathbf{r}_4)}. \tag{2}$$

Although these equations were derived before the development of adaptive optics[3] and the wave-front distortion considered then was the uncorrected wave-front distortion (i.e., wave-front distortion without any adaptive optics correction), those results are fully applicable to an adaptive optics system if the measure of the wave-front-distortion statistics inherent in the F_1 and F_2 functions appearing in Eqs. (1) and (2) is understood to be that for the residual wave-front distortion after adaptive optics correction.

The function $W(\mathbf{r})$ denotes the extent of the system's aperture, having a value of unity for a position $\mathbf{r}$ that is inside the aperture and a value of zero for a position that is outside the aperture. We have considered the system that has a clear circular aperture of diameter D, for which case we can write

$$W(\mathbf{r}) = \begin{cases} 1 & \text{if } |r| \leq D/2 \\ 0 & \text{if } |r| > D/2 \end{cases}. \tag{3}$$

The arguments of the exponential functions in Eqs. (1) and (2), denoted by F_1 and F_2, have values given by the equations

$$F_1 = -\tfrac{1}{2}[\mathcal{D}_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) + \mathcal{D}_l(\mathbf{r}_1 - \mathbf{r}_2)] \tag{4}$$

$$\begin{aligned} F_2 = -\tfrac{1}{2}[&\mathcal{D}_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) - \mathcal{D}_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_3) + \mathcal{D}_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_4) \\ &+ \mathcal{D}_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_3) - \mathcal{D}_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_4) \\ &+ \mathcal{D}_{\delta\phi}(\mathbf{r}_3 - \mathbf{r}_4) + \mathcal{D}_l(\mathbf{r}_1 - \mathbf{r}_2) - \mathcal{D}_l(\mathbf{r}_1 - \mathbf{r}_3) \\ &+ \mathcal{D}_l(\mathbf{r}_1 - \mathbf{r}_4) + \mathcal{D}_l(\mathbf{r}_2 - \mathbf{r}_3) - \mathcal{D}_l(\mathbf{r}_2 - \mathbf{r}_4) \\ &+ \mathcal{D}_l(\mathbf{r}_3 - \mathbf{r}_4)] + 2[\mathcal{C}_l(\mathbf{r}_1 - \mathbf{r}_3) + \mathcal{C}_l(\mathbf{r}_2 - \mathbf{r}_4)]. \end{aligned} \tag{5}$$

The notations $\mathcal{D}_{\delta\phi}(\mathbf{r})$ and $\mathcal{D}_l(\mathbf{r})$ denote the structure functions for the (residual) phase and for the log amplitude. The notations $\mathcal{C}_{\delta\phi}(\mathbf{r})$ and $\mathcal{C}_l(\mathbf{r})$ denote the covariance of the residual phase and of the log amplitude, respectively.

The structure functions and the covariance functions are related by the equations $\mathcal{D}_{\delta\phi}(\mathbf{r}) = 2[\mathcal{C}_{\delta\phi}(\mathbf{0}) - \mathcal{C}_{\delta\phi}(\mathbf{r})]$ and $\mathcal{D}_l(\mathbf{r}) = 2[\mathcal{C}_l(\mathbf{0}) - \mathcal{C}_l(\mathbf{r})]$, and the variances and the covariance functions are related by the equations $\sigma_{\delta\phi}^2 = \mathcal{C}_{\delta\phi}(\mathbf{0})$ and $\sigma_l^2 = \mathcal{C}_l(\mathbf{0})$. We can express the structure functions as

$$\mathcal{D}_{\delta\phi}(\mathbf{r}) = 2\sigma_{\delta\phi}^2[1 - f_{\delta\phi}(\mathbf{r})], \quad \text{where } f_{\delta\phi}(\mathbf{r}) = \mathcal{C}_{\delta\phi}(\mathbf{r})/\sigma_{\delta\phi}^2, \tag{6}$$

and

$$\mathcal{D}_l(\mathbf{r}) = 2\sigma_l^2[1 - f_l(\mathbf{r})], \qquad \text{where } f_l(\mathbf{r}) = \mathcal{C}_l(\mathbf{r})/\sigma_l^2. \tag{7}$$

With this notation it is convenient to recast Eqs. (4) and (5) as

$$F_1 = \sigma_{\delta\phi}^2[-1 + f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)] + \sigma_l^2[-1 + f_l(\mathbf{r}_1 - \mathbf{r}_2)], \tag{8}$$

$$\begin{aligned} F_2 = \sigma_{\delta\phi}^2[&-2 + f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) - f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_3) \\ &+ f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_4) + f_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_3) - f_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_4) \\ &+ f_{\delta\phi}(\mathbf{r}_3 - \mathbf{r}_4)] + \sigma_l^2[-2 + f_l(\mathbf{r}_1 - \mathbf{r}_2) \\ &+ f_l(\mathbf{r}_1 - \mathbf{r}_3) + f_l(\mathbf{r}_1 - \mathbf{r}_4) + f_l(\mathbf{r}_2 - \mathbf{r}_3) \\ &+ f_l(\mathbf{r}_2 - \mathbf{r}_4) + f_l(\mathbf{r}_3 - \mathbf{r}_4)]. \end{aligned} \tag{9}$$

Based on the assumption that the variance of the residual phase, $\sigma_{\delta\phi}^2$, and the log-amplitude variance σ_l^2 are both small, and recognizing that $f_{\delta\phi}(\mathbf{r})$ and $f_l(\mathbf{r})$ have magnitudes no greater than unity, we consider the power series expansion of the exponential functions in Eqs. (1) and (2), retaining only terms up to second order in the argument of the exponential, i.e., using the approximation that $\exp(x) \approx 1 + x + \frac{1}{2}x^2$. With this approximation we obtain the results that

$$\begin{aligned} \exp(F_1) \approx 1 &+ \{\sigma_{\delta\phi}^2[-1 + f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)] \\ &+ \sigma_l^2[-1 + f_l(\mathbf{r}_1 - \mathbf{r}_2)]\} \\ &+ \tfrac{1}{2}((\sigma_{\delta\phi}^2)^2\{1 - 2f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) \\ &+ [f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)]^2\} + (\sigma_l^2)^2\{1 - 2f_l(\mathbf{r}_1 - \mathbf{r}_2) \\ &+ [f_l(\mathbf{r}_1 - \mathbf{r}_2)]^2\} + 2\sigma_{\delta\phi}^2\sigma_l^2[1 - f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) \\ &- f_l(\mathbf{r}_1 - \mathbf{r}_2) + f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_1 - \mathbf{r}_2)]) \end{aligned} \tag{10}$$

and (after considerable algebraic manipulation) the result that

$$\begin{aligned} \exp(F_2) \Rightarrow 1 &+ \{\sigma_{\delta\phi}^2[-2 + 2f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)] \\ &+ \sigma_l^2[-2 + 6f_l(\mathbf{r}_1 - \mathbf{r}_2)]\} \\ &+ \tfrac{1}{2}\{(\sigma_{\delta\phi}^2)^2[4 - 8f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) + K_{\delta\phi}] \\ &+ (\sigma_l^2)^2[4 - 24f_l(\mathbf{r}_1 - \mathbf{r}_2) + K_l] \\ &+ 2\sigma_{\delta\phi}^2\sigma_l^2[4 - 4f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) \\ &- 12f_l(\mathbf{r}_1 - \mathbf{r}_2) + K_{\delta\phi,l}]\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} K_{\delta\phi} = 6f_{\delta\phi}&(\mathbf{r}_1 - \mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) \\ &- 8f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_3) \\ &+ 6f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_3 - \mathbf{r}_4), \end{aligned} \tag{12a}$$

$$\begin{aligned} K_l = 6f_l(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_1 - \mathbf{r}_2) + 24f_l(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_2 - \mathbf{r}_3) \\ + 6f_l(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_3 - \mathbf{r}_4), \end{aligned} \tag{12b}$$

$$\begin{aligned} K_{\delta\phi,l} = 2f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_1 - \mathbf{r}_2) + 8f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2) \\ \times f_l(\mathbf{r}_2 - \mathbf{r}_3) + 2f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_3 - \mathbf{r}_4). \end{aligned} \tag{12c}$$

When relations (10) and (11) are substituted into Eqs. (1) and (2), we can obtain the results that

$$\begin{aligned} \langle S \rangle = 1 &+ \sigma_{\delta\phi}^2(-1 + \alpha_{\delta\phi}) + \sigma_l^2(-1 + \alpha_l) \\ &+ \tfrac{1}{2}(\sigma_{\delta\phi}^2)^2(1 - 2\alpha_{\delta\phi} + \beta_{\delta\phi}) \\ &+ \tfrac{1}{2}(\sigma_l^2)^2(1 - 2\alpha_l + \beta_l) \\ &+ \sigma_{\delta\phi}^2\sigma_l^2(1 - \alpha_{\delta\phi} - \alpha_l + \beta_{\phi,l}), \end{aligned} \tag{13}$$

and

$$\langle S^2\rangle = 1 + \sigma_{\delta\phi}^2(-2 + 2\alpha_{\delta\phi}) + \sigma_l^2(-2 + 6\alpha_l) + (\sigma_{\delta\phi}^2)^2(2 - 4\alpha_{\delta\phi} + 3\beta_{\delta\phi} - 4\gamma_{\delta\phi} + 3\alpha_{\delta\phi}^2) + (\sigma_l^2)^2(2 - 12\alpha_l + 3\beta_l + 12\gamma_l + 3\alpha_l^2) + \sigma_{\delta\phi}^2\sigma_l^2(4 - 4\alpha_{\delta\phi} - 12\alpha_l + 2\beta_{\phi,l} + 8\gamma_{\phi,l} + 2\alpha_{\delta\phi}\alpha_l), \tag{14}$$

where

$$\alpha_{\delta\phi} = \frac{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)}{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{15a}$$

$$\alpha_l = \frac{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)f_l(\mathbf{r}_1 - \mathbf{r}_2)}{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{15b}$$

$$\beta_{\delta\phi} = \frac{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)[f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)]^2}{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{16a}$$

$$\beta_l = \frac{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)[f_l(\mathbf{r}_1 - \mathbf{r}_2)]^2}{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{16b}$$

$$\beta_{\delta\phi,l} = \frac{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_1 - \mathbf{r}_2)}{\iint \mathbf{dr}_1\mathbf{dr}_2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \tag{16c}$$

and

$$\gamma_{\delta\phi} = \frac{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_{\delta\phi}(\mathbf{r}_2 - \mathbf{r}_3)}{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)}, \tag{17a}$$

$$\gamma_l = \frac{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)f_l(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_2 - \mathbf{r}_3)}{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)}, \tag{17b}$$

$$\gamma_{\delta\phi,l} = \frac{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)f_{\delta\phi}(\mathbf{r}_1 - \mathbf{r}_2)f_l(\mathbf{r}_2 - \mathbf{r}_3)}{\iiint \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3 W(\mathbf{r}_1)W(\mathbf{r}_2)W(\mathbf{r}_3)}. \tag{17c}$$

The variance of the Strehl ratio, σ_S^2, is expressed in terms of the first and second moments of the Strehl ratio, $\langle S\rangle$ and $\langle S^2\rangle$, respectively, by the equation $\sigma_S^2 = \langle S^2\rangle - \langle S\rangle^2$. Making use of Eq. (13) and after carrying out the squaring process and dropping terms of too-high order, we obtain the result that

$$\langle S\rangle^2 = 1 + \sigma_{\delta\phi}^2(-2 + 2\alpha_{\delta\phi}) + \sigma_l^2(-2 + 2\alpha_l) + (\sigma_{\delta\phi}^2)^2(2 - 4\alpha_{\delta\phi} + \alpha_{\delta\phi}^2 + \beta_{\delta\phi}) + (\sigma_l^2)^2(2 - 4\alpha_l + \alpha_l^2 + \beta_l) + \sigma_{\delta\phi}^2\sigma_l^2(4 - 4\alpha_{\delta\phi} - 4\alpha_l + 2\alpha_{\delta\phi}\alpha_l + 2\beta_{\delta\phi,l}). \tag{18}$$

This result together with Eq. (14) allows us to write for σ_S^2, the variance of the Strehl ratio, that

$$\sigma_S^2 = \sigma_l^2(4\alpha_l) + (\sigma_{\delta\phi}^2)^2(2\beta_{\delta\phi} + 2\alpha_{\delta\phi}^2 - 4\gamma_\phi) + (\sigma_l^2)^2(-8\alpha_l + 2\beta_l + 2\alpha_l^2 + 12\gamma_l) + \sigma_{\delta\phi}^2\sigma_l^2(-8\alpha_l + 8\gamma_{\delta\phi,l}). \tag{19}$$

Considering this result, we can see that the variance of the Strehl ratio is first order in the log-amplitude variance but only second order in the variance of the residual phase. We can consider this our final result, or if we wish, in the spirit of order-of-magnitude approximations, we can make the approximations that

$$\alpha_{\delta\phi} \approx (d/D)^2, \quad \alpha_l \approx (\rho/D)^{7/3}, \tag{20a}$$

$$\beta_{\delta\phi} \approx (d/D)^2, \quad \beta_l \approx (\rho/D)^{7/3}, \quad \beta_{\delta\phi,l} \approx (d/D)^2, \tag{20b}$$

$$\gamma_{\delta\phi} \approx (d/D)^4, \quad \gamma_l \approx (\rho/D)^{14/3}, \quad \gamma_{\delta\phi,l} \approx (d/D)^2(\rho/D)^{7/3}, \tag{20c}$$

where d denotes the adaptive optics subaperture size and may be considered to be the correlation length for the residual phase, and ρ denotes the correlation length for the log-amplitude variations. These approximations follow from consideration of Eqs. (6) and (7) and Eqs. (15)–(17)

and from consideration of the facts that the value of $f_{\delta\phi}(\mathbf{r})$ goes to zero rather rapidly for values of $\mathbf{r}$ for which $|\mathbf{r}| > d$, that the value of $f_l(\mathbf{r})$ goes to zero (some what more slowly) when the value of $\mathbf{r}$ is such that $|\mathbf{r}| > \rho$, that for all adaptive optics systems $d \ll D$, and that the value of ρ is of the order of a few to a few tens of centimeters (depending on the wavelength), while the telescope aperture diameter D is of the order of meters—so $\rho \ll D$. That the ρ over D ratio shows up in relations (20) raised to the seven-thirds power rather than to the second (or six-thirds) power is a consequence of the way in which $f_l(\mathbf{r})$ approaches zero as the value of $|\mathbf{r}|$ increases—with an initial undershoot (i.e., a drop below zero value)—as developed by Gracheva and Gurvich[4] and by Lutomirski *et al.*[5] In writing the expression for $\beta_{\delta\phi,l}$ we have made use of the fact that (except under conditions of saturated scintillation[6]) ρ will be greater than r_0 and thus greater than d, so the effective range of the integrand in Eq. (16c) is set by d and not by ρ.

The approximation sign is used in relations (20) to indicate only an order-of-magnitude dependence. In this order-of-magnitude sense we can recast Eq. (19) as

$$\sigma_S^2 \approx \sigma_l^2(\rho/D)^{7/3} + (\sigma_{\delta\phi}^2)^2(d/D)^2 + (\sigma_l^2)^2(\rho/D)^{7/3} + \sigma_{\delta\phi}^2\sigma_l^2(\rho/D)^{7/3}. \tag{21}$$

The first thing to be noted in considering Eq. (21) is that in all existing adaptive optics systems $d \ll D$ and that for all propagation conditions of interest the value of ρ is comparable to or larger than r_0 and is thus comparable to or larger than d.

This allows us to conclude that to leading order, the Strehl ratio variance, σ_S^2 does not depend on the variance of the residual phase, $\sigma_{\delta\phi}^2$, but rather on the log-amplitude variance σ_l^2. Only in the extreme case where the scintillation is so weak that σ_l^2 is less than $(\sigma_{\delta\phi}^2)^2(d/\rho)^2$ would the $(\sigma_{\delta\phi}^2)^2$-dependent term be dominant. The general form of the order-of-magnitude result can be written as

$$\sigma_S^2 \approx \begin{cases} \sigma_l^2(\rho/D)^{7/3} & \text{if } \sigma_l^2 \geq (\sigma_{\delta\phi}^2)^2(d/\rho)^2(D/\rho)^{1/3} \\ (\sigma_{\delta\phi}^2)^2(d/D)^2 & \text{otherwise} \end{cases}. \tag{22}$$

This presumes for its development that $\sigma_{\delta\phi}^2 < 1$.

Starting from Eq. (13), the corresponding results for the mean value of the Strehl ratio, $\langle S\rangle$, is given by the relation

$$\langle S\rangle \approx 1 - \sigma_{\delta\phi}^2 - \sigma_l^2, \tag{23}$$

just as we might have expected. The leading value in the deviation from unity of the mean value of the Strehl ratio can be either $\sigma_{\delta\phi}^2$ or σ_l^2 depending on which is greater. The deviation from unity of the mean value of the Strehl ratio can be dominated by the mean square residual phase error, whereas the variance of the Strehl ratio cannot.

From consideration of relation (23) we can see that $\langle S\rangle$, the mean value of the Strehl ratio, is dependent on the first power of $\sigma_{\delta\phi}^2$, the mean square value of the residual phase error. The effect of σ_l^2, the log-amplitude variance, which generally is significantly smaller than $\sigma_{\delta\phi}^2$, is consequently generally only of secondary importance in determining the value of $\langle S\rangle$. On the other hand, we see from consideration of Eqs. (21) and (22) that for σ_S^2, the variance of the Strehl ratio, generally the dependence is principally on σ_l^2, the log-amplitude variance, with only a second-order dependence on $\sigma_{\delta\phi}^2$, the mean square value of the residual phase error—even though the value of $\sigma_{\delta\phi}^2$ can be significantly larger than the value of σ_l^2.

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REFERENCES AND NOTES

1. In regard to the matter of an exact definition for resolution, it may be noted that if the concept of resolution is taken to be represented by the value of the optical transfer function integrated over the spatial frequency domain, then resolution as thus defined is directly proportional to the Strehl ratio.
2. It is perhaps worth noting that in the matter of resolution the variations of intensity over the telescope's aperture will affect the image of a point source, i.e., will affect the resolution. That this is indeed the case is perhaps made evident most easily by simply referring to the intensity variations as "random apodization" of the telescope.

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